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# Envelope solitons and holes for sine–Gordon and non-linear Klein–Gordon equations

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**Abstract.** The non-linear Schrödinger equations governing the evolution of the plane wave solutions of the sine–Gordon and the non-linear Klein–Gordon equations are derived. The former is unstable against modulational instability and thus evolves into envelope solitons. The latter however can be unstable or stable depending on the sign of the cubic term and can accordingly admit envelope soliton or hole solutions.

## 1. Introduction

The evolution of a variety of physical systems is governed by the equation

$$\Phi_{tt} - \Phi_{xx} + V'(\Phi) = 0, \tag{1}$$

where $V'(\Phi)$ is a non-linear function of $\Phi$ and may be taken as the derivative of a potential energy $V(\Phi)$ (Barone *et al* 1971, Whitham 1974). For $V(\Phi) = -\cos\Phi$, equation (1) becomes the sine–Gordon (SG) equation, namely

$$\Phi_{tt} - \Phi_{xx} + \sin\Phi = 0, \tag{2}$$

and when $V(\Phi) = \Phi^2/2 + \alpha\Phi^4/4$, we get the non-linear Klein–Gordon (NKG) equation,

$$\Phi_{tt} - \Phi_{xx} + \Phi + \alpha\Phi^3 = 0. \tag{3}$$

In equations (2) and (3) the variables $\Phi$, $x$ and $t$, and the constant $\alpha$ have been made dimensionless by appropriate normalizations. Equation (3) corresponds to a small amplitude expansion of equation (2) when $\alpha = -1/6$. This equation describes the many-body behaviour of elementary particles (Schiff 1951). Many physical systems represented by equation (2) are described by Barone *et al* (1971) and Whitham (1974). This equation arises in the study of Josephson junctions, dislocations in crystals, ferromagnetic materials, laser pulse propagation, etc. Also based on equation (2), Perring and Skyrme (1962) have discussed the strong interaction among elementary particles. A lucid discussion of the relevance of equations (2) and (3) to quantum field theory is given in the recent review by Rajaraman (1975). The modulational instability of equation (3) has been studied by Asano *et al* (1969) by using the reductive perturbation method (Taniuti and Yajima 1969).

Here we study the modulational instability and then obtain the localized stationary solutions of equations (2) and (3). For this purpose we use the multiple space–time method (Kakutani and Sugimoto 1974, Buti 1976, Sharma and Buti 1976) to obtain the non-linear Schrödinger (NLS) equations which describe the slow variations of the amplitudes of the plane wave solutions of these equations.

## 2. Sine–Gordon equation

In order to study the envelope properties of the SG equation, we use the transformation $\phi = \tan(\Phi/4)$; equation (2) then reduces to

$$(1+\phi^2)(\phi_{tt}-\phi_{xx}+\phi)-2\phi(\phi_t^2-\phi_x^2+\phi^2)=0. \tag{4}$$

The NLS equation describing the envelope of the plane wave solutions of this equation can be derived as follows.

Let us consider a perturbation solution of equation (4) of the form

$$\phi = \epsilon\phi_1(a, \bar{a}, \psi)+\epsilon^2\phi_2(a, \bar{a}, \psi)+ \dots, \tag{5}$$

where $a$ is the complex amplitude and $\psi = kx-\omega t$ is the phase factor ($\bar{a}$ is the complex conjugate of $a$). The scalar $\phi$ is a real function of $x$ and $t$ only through $a$, $\bar{a}$ and $\psi$. The slow variations of the amplitudes are defined by

$$\frac{\partial a}{\partial t}=\epsilon A_1+\epsilon^2 A_2+\dots \qquad \text{and} \qquad \frac{\partial a}{\partial x}=\epsilon B_1+\epsilon^2 B_2+\dots .$$

To lowest order (order $\epsilon$), equation (4) has the plane wave solution given by

$$\phi_1 = a\exp(\mathrm{i}\psi)+\mathrm{cc},$$

where cc represents the complex conjugate, $\omega$ and $k$ satisfy the linear dispersion relation

$$\mathrm{D}(\omega, k)\equiv -\omega^2+k^2+1=0. \tag{6}$$

The equation to order $\epsilon^2$ is

$$\omega^2\frac{\partial^2\phi_2}{\partial\psi^2}-k^2\frac{\partial^2\phi_2}{\partial\psi^2}+\phi_2=2\mathrm{i}\omega\left(A_1+\frac{k}{\omega}B_1\right)\exp(\mathrm{i}\psi)+\mathrm{cc}. \tag{7}$$

The terms on the right-hand side give rise to resonant secularity and this is removed by the condition

$$A_1+V_{\mathrm{g}}B_1=0$$

where $V_{\mathrm{g}}=k/\omega$ is the group velocity. The secular free solution of equation (7) is then given by

$$\phi_2=b(a, \bar{a})\exp(\mathrm{i}\psi)+\mathrm{cc}$$

where $b(a, \bar{a})$ is a function of $a$ and $\bar{a}$, but a constant with respect to $\psi$. Equation (4) to order $\epsilon^3$ may similarly be written down; the condition for the removal of the resonant singularity in this case is simply

$$\mathrm{i}(A_2+V_{\mathrm{g}}B_2)+\frac{1}{2}\frac{\mathrm{d}V_{\mathrm{g}}}{\mathrm{d}k}\left(B_1\frac{\partial B_1}{\partial a}+\bar{B}_1\frac{\partial B_1}{\partial \bar{a}}\right)+\frac{4}{\omega}|a|^2 a=0. \tag{8}$$

From the definitions we know that

$$A_2 = \frac{\partial a}{\partial t_2} - \frac{A_1}{\epsilon}, \qquad B_2 = \frac{\partial a}{\partial x_2} - \frac{B_1}{\epsilon} \qquad B_1 \frac{\partial B_1}{\partial a} + \bar{B}_1 \frac{\partial B_1}{\partial \bar{a}} = \frac{\partial^2 a}{\partial x_1^2},$$

where $t_2 = \epsilon^2 t$, $x_2 = \epsilon^2 x$ and $x_1 = \epsilon x$. Equation (8) can thus be rewritten as

$$\mathrm{i}\left(\frac{\partial a}{\partial t_2} + V_g \frac{\partial a}{\partial x_2}\right) + \frac{1}{2}\frac{\mathrm{d}V_g}{\mathrm{d}k}\frac{\partial^2 a}{\partial x_1^2} + \frac{4}{\omega}|a|^2 a = 0,$$

which with a coordinate transformation

$$\xi = \frac{1}{\epsilon}(x_2 - V_g t_2) = x_1 - V_g t_1 = \epsilon(x - V_g t),$$

$$\tau = t_2 = \epsilon t_1 = \epsilon^2 t,$$

reduces to

$$\mathrm{i}\frac{\partial a}{\partial \tau} + p\frac{\partial^2 a}{\partial \xi^2} + q|a|^2 a = 0, \tag{9}$$

where

$$p = \frac{1}{2}\frac{\mathrm{d}V_g}{\mathrm{d}k} = \frac{1}{2\omega^3} \qquad \text{and} \qquad q = \frac{4}{\omega}. \tag{10}$$

Equation (9) is the non-linear Schrödinger equation governing the envelope properties of the plane wave solutions of equation (4).

To investigate the stability of equation (9) against long wavelength perturbations we take (Hasegawa 1975):

$$a = \rho^{1/2}(\xi, \tau)\exp(\mathrm{i}\sigma(\xi, \tau)),$$

so that it can be separated into real and imaginary parts. On linearizing the resulting equations as

$$\begin{pmatrix}\rho \\ \sigma\end{pmatrix} = \begin{pmatrix}\rho_0 \\ \sigma_0\end{pmatrix} + \begin{pmatrix}\rho_1 \\ \sigma_1\end{pmatrix}\exp[\mathrm{i}(K\xi - \Omega\tau)],$$

and since according to equation (10), $pq = 2/\omega^4 > 0$ we find that the perturbations with $K < (4\omega\rho_0^{1/2})$ are unstable. These equations have the localized stationary solutions

$$\rho = \rho_s \operatorname{sech}^2[(q\rho_s/2p)^{1/2}\xi], \tag{11}$$

where $\rho_s$ is a constant and $p$, $q$ are given by equation (10). This is an envelope soliton.

## 3. Non-linear Klein–Gordon equation

The NLS equation describing the envelope of the plane wave solutions of the NKG equation (equation (3)), can be obtained in the same manner as above by using the multiple space–time method. On considering a perturbation expansion of the form of equation (5), we find that the plane wave solutions of equation (3) satisfy the dispersion relation given by equation (6). The condition for the removal of the resonant secularity in equation (3) to order $\epsilon^3$ yields the NLS equation, given by equation (9), with the coefficients $p$ and $q$ given by

$$p = \frac{1}{2\omega^3}, \qquad q = -\frac{3\alpha}{2\omega} \qquad \text{and} \qquad pq = -\frac{3\alpha}{4\omega^3}. \tag{12}$$

These coefficients are identical to the ones obtained by Asano *et al* (1969). If $\alpha < 0$, the plane wave solutions of the NKG equation are modulationally unstable and have envelope soliton solutions, as given by equation (11). On the other hand if $\alpha > 0$, these plane waves are modulationally stable and have the envelope hole solutions of the form

$$\rho = \rho_1\{1 - \tilde{a}^2 \operatorname{sech}^2[(|pq|\rho_1/2p^2)^{1/2}\tilde{a}\xi]\},$$

where $p, q$ are as defined in equation (12), $\rho_1$ is the asymptotic value of the wave amplitude and $\tilde{a}$ the depth of the modulation with respect to this value (Hasegawa 1975). This solution represents a depression, i.e., a region characterized by the absence of the wave intensity, propagating with the group velocity $V_g$.

As pointed out earlier, equation (3) corresponds to the small amplitude expansion of equation (2) when $\alpha = -1/6$. On putting this value of $\alpha$ in equation (12) we get $pq = 1/(8\omega^4) > 0$ and thus under this approximation also the plane wave solutions of the SG equation are modulationally unstable and hence the equation admits envelope solitons.

Here we have studied the envelope properties of the sine–Gordon equation by transforming it into the NLS equation which admits envelope solitons. The envelope solitons discussed here are the bound states of $n(=k/K)$ quasiparticles represented by the plane waves in contrast to the two kink or soliton bound states which have been given various names: 'mesons' in the non-linear field theory of elementary particles (Perring and Skyrme 1962), '$0\pi$' pulses in the self-induced transparency problems (Lamb 1971), 'bions' (Caudrey *et al* 1973) and 'doublets' (Rajaraman 1975). In comparison with the doublets, these envelope solitons may be called 'multiplets'. In the non-linear field theory of elementary particles the kinks correspond to extended or dressed particles and the mesons or bions or doublets are the bound states of two such particles. The constituents of the multiplets in our case are the plane waves which unlike the kinks do not carry the effect of the non-linearity and hence may correspond to 'bare' particles.

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